MATH 642:491–Fall, 2024

**Problems on Real Analysis**

1. Let $a, b, c$ be arbitrary real numbers. Show that there is a number $x \in [0,1]$ such that $4ax^3 + 3bx^2 + 2cx = a + b + c$.

2. Suppose that $a_0, \ldots, a_n$ are real numbers such that $\sum_{i=0}^{n} \frac{a_i}{i+1} = 0$. Prove that the polynomial $\sum_{i=0}^{n} a_i x^i$ has a real root.

3. Prove that $x^2 = xsin(x) + cos(x)$ for exactly two real numbers $x$.

4. Suppose $f : \mathbb{R} \to R$ is a $n$-times differentiable function, and suppose $a_1 < a_2 < \ldots < a_{n+1}$ are such that $f(a_i) = 0$ for each $i \in \{1, \ldots, n+1\}$. Show that there is some $b \in [a_1, a_{n+1}]$ such that $f^{(n)}(b) = 0$.

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that for every $a, b$, if $\frac{f(a)+f(b)}{2} = f\left(\frac{a+b}{2}\right)$. Show that $f(x) = \alpha x + \beta$ for some $\alpha, \beta$.

   If we drop the condition that $f$ is continous, does the same conclusion hold?

6. Suppose $f$ is a differentiable function defined on $[0,1]$ and $f(0) = 0$ and $f(x) > 0$ for $x \in (0,1]$. Prove that there is a number $c \in (0,1)$ such that

$$\frac{2f'(c)}{f(c)} - \frac{f'(1-c)}{f(1-c)}$$

7. Show that there is a unique real number $c$ such that for every differentiable function $f : [0,1] \to \mathbb{R}$ with $f(0) = 0$ and $f(1) = 1$, the equation $f'(x) = cx$ has a solution.

8. Find all continuous functions $f : \mathbb{R}_{>1} \longrightarrow \mathbb{R}_{>0}$ such that for every $x > 1$,

$$\int_1^x f(t)dt = \int_x^{x^2} f(t)dt.$$

9. Let $f : [0,1] \longrightarrow \mathbb{R}$ be continuous with $f(0) = f(1)$. Let $t \geq 1$. Prove that if $t$ is an integer then there is an $x \in [0, 1 - 1/t]$ such that $f(x) = f(x + \frac{1}{t})$. Prove that this statement is not true in general if $t$ is not an integer.

10. Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable for all $x$. Suppose $f(0) = 0$ and $|f'(x)| \leq |f(x)|$ for all $x$. Prove that $f(x) = 0$ for all $x$.

11. Given two nonintersecting convex polygons in the plane, show that there is a line that splits each of them into two regions of equal area.

12. Suppose $n$ is a positive integer and $c_0, \ldots, c_n$ and $r_0, \ldots, r_n$ are real numbers and let $f$ be the function $f(x) = \sum_{i=0}^{n} c_i e^{r_i x}$. Prove that if $f$ has $n+1$ zeroes then it is identically 0.

13. (*) Let $f(x)$ be a differentiable function on $[0,1]$ with $f(0) = 0$ and $f(1) = 1$. Show that for every positive integer $n$ there are distinct points $x_1 < \cdots < x_n$ such that $\sum_{i=1}^{n} \frac{1}{f'(x_i)} = n$.

14. (Putnam) Find $\lim_{t\to\infty} e^{-t}\int_0^t\int_0^t \frac{e^x-e^y}{x-y}dydx$, or determine that the limit does not exist.

Useful facts to recall.

Suppose $f : [a, b] \longrightarrow \mathbb{R}$.

- **Extreme Value Theorem.** If $f$ is continuous then there exist $c, d \in [a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$.
- **Intermediate Value Theorem.** If $f$ is continuous then for $y \in [\min(f(a), f(b)), \max(f(a), f(b)]$ there is $x \in [a, b]$ such that $f(x) = y$.
- **Mean Value Theorem.** If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$ then there is a number $c \in (a, b)$ such that $f'(c) = (f(b) - f(a))/(b - a)$.